

May 31, 1995

TAUP-2247-95

hep-th/9506002

The other topological twisting of $N = 4$ Yang–Mills

Neil Marcus*

School of Physics and Astronomy

Raymond and Beverly Sackler Faculty of Exact Sciences

Tel-Aviv University

Ramat Aviv, Tel-Aviv 69978, ISRAEL.

Abstract

We present the alternative topological twisting of $N = 4$ Yang–Mills, in which the path integral is dominated not by instantons, but by flat connections of the *complexified* gauge group. The theory is nontrivial on compact orientable four-manifolds with nonpositive Euler number, which are necessarily not simply connected. On such manifolds, one finds a single topological invariant, analogous to the Casson invariant of three-manifolds.

*E-Mail: NEIL@POST.TAU.AC.IL.

Work supported in part by the US-Israel Binational Science Foundation, the German-Israeli Foundation for Scientific Research and Development and the Israel Academy of Science.

1 Introduction

One of the simplest ways of making a topological field theory is to take a theory with an extended spacetime supersymmetry in Euclidean space and to “twist it”, changing the spins of the fields [1, 2]. The resulting theory will be topological if the twisted supersymmetry generators include at least one spacetime scalar, which can then be thought of as a BRST operator. The prototypical example of this is Witten’s twisting of pure $N = 2$ Yang–Mills [1]. $N = 2$ supersymmetry has a $U(2)$ R–parity group; the twisting involves identifying its $SU(2)$ with one of the $SU(2)$ ’s of the spacetime Lorentz group, and interpreting its $U(1)$ as an anomalous “ghost number” symmetry. Under this twisting, one of the fermions of the theory gives rise to a self-dual antisymmetric 2–tensor $\chi_{\mu\nu}^+$, whose BRST transformation is the self-dual part of the Yang–Mills field strength $F_{\mu\nu}^+$. Since topological theories are controlled by fixed points of their BRST transformations [3], the path integral is dominated by (anti)-instanton configurations. Witten studied the cohomology of this theory, and showed that the amplitudes of these operators calculate the Donaldson invariants of the underlying manifold [4].

In the $N = 2$ case the twisting procedure is unique, so the most general topological theory is formed by twisting $N = 2$ Yang–Mills coupled to $N = 2$ matter hypermultiplets in various representations of the gauge group [5, 6]. Such theories have the same cohomology as the Witten–Donaldson (WD) theory, but their ghost-number anomalies, ground states, and amplitudes are changed. To have a fundamentally different theory, one needs to increase the number of supersymmetries. Since topological gravity does not come from twisting supergravity theories, the only possibility is to start from $N = 4$ Yang–Mills. This theory is uniquely specified by its gauge group, but its $SU(4)$ R–parity group can be twisted in three different ways, leading to three distinct topological theories [7].

The lagrangians of the first two theories were written down by Yamron, using superspace techniques [5]. The first—which we shall call the “half-twisted” theory—involves simply treating the $N = 4$ Yang–Mills as an $N = 2$ theory. It is thus a particular (but the first, and maybe an especially interesting) example of a WD theory with matter. Explicitly, the twisting is carried out by breaking the $SU(4)$ to an $SU(2) \otimes SU(2) \otimes U(1)$, with the $\mathbf{4} \rightarrow (2, 1)^1 \oplus (1, 2)^{-1}$, and identifying one of the $SU(2)$ ’s with a spacetime $SU(2)$. The other $SU(2)$ remains an internal symmetry of the theory.

The second twisting—giving rise to what we shall call the Yamron–Vafa–Witten (YVW) theory—is found by reducing the $SU(4)$ to an $SO(4)$, with the $\mathbf{4} \rightarrow \mathbf{4}$, and identifying one of the resulting $SU(2)$ ’s with a spacetime $SU(2)$. This theory thus has an “ $SU(2)$ ghost-number” symmetry, and *two* BRST symmetries, transforming as a doublet

of this $SU(2)$ [5]. There is also a doublet of $\chi_{\mu\nu}^+$ fields, which both transform into $F_{\mu\nu}^+$, so the theory is again dominated by instantons. Because such a nonabelian ghost number can not be anomalous, the partition function is nonvanishing, and indeed is the only observable of the theory. Vafa and Witten showed that under appropriate conditions it gives the Euler number of the moduli space of instantons, and that it transforms covariantly under certain S–duality transformations [7].

The existence of the third twisting was pointed out in [5], as a private communication from E. Witten, and again in [7]. This theory has not been explicitly constructed, and it shall be the interest of this paper. It can be obtained by further twisting the internal $SU(2)$ of the half-twisted theory* with the remaining spacetime $SU(2)$, and differs basically from the $N = 2$ and the YVW theories. The theory has a $U(1)$ ghost-number symmetry, and two BRST operators, both with ghost number +1. These are interchanged by a hermitian conjugation operation, unlike in the YVW theory, where the two BRST charges are equivalent.

The fields of the twisted theory are:

$N = 4$ YM	\rightarrow	twisted theory	dimension	ghost number
A_μ	\rightarrow	A_μ	1	0
$\Psi^I, \Psi_I^{(c)}$	\rightarrow	$\chi_{\mu\nu}$	3/2	–1
	\rightarrow	$\psi_\mu, \tilde{\psi}_\mu$	3/2	1
	\rightarrow	$\eta, \tilde{\eta}$	3/2	–1
Φ_{IJ}	\rightarrow	V_μ	1	0
	\rightarrow	B	1	2
	\rightarrow	C	1	–2
nothing	\rightarrow	P	2	0

They are all in the adjoint representation of the gauge group. (The twisted anticommuting fields actually correspond to sums and differences of Ψ^I and $\Psi_I^{(c)}$. For example, Ψ^I twists to give the anti-self-dual part of $\chi_{\mu\nu}$.) Inspired by the half-twisted theory [5], we have introduced an auxiliary field P , in order to help close the BRST algebra†. As

*Thus justifying our nomenclature. In some ways, the three theories are analogous to the half-twisted, A and B $N = 2$ topological sigma models in two dimensions [8].

†One would also have expected to have at least one auxiliary antisymmetric tensor field $B_{\mu\nu}$ in analogy with the other twistings [5]. We have been unable to find the appropriate auxiliary fields in our case, and this will cause some technical nuisances later in the paper.

in the WD [1, 5] theories, one can redefine a “dimension” to be the dimension minus half the ghost number. This results in dimensionless BRST charges, but has little practical advantage.

There are several points worth noting about this spectrum:

- **It is not chiral.**

- **Hermitian-conjugation:**

One can define an antihermitian “hermitian-conjugation” transformation, coming from the CP invariance of the original $N = 4$ Yang–Mills. Under this conjugation ψ_μ and η are complex, transforming into $\tilde{\psi}_\mu$ and $\tilde{\eta}$; A_μ, V_μ, B, C, P and the self-dual piece of $\chi_{\mu\nu}$ are real, and the anti-self-dual piece of $\chi_{\mu\nu}$ is purely imaginary:

$$\begin{aligned} (A_\mu, V_\mu, B, C, P) &\rightarrow -(A_\mu, V_\mu, B, C, P) \\ (\psi_\mu, \eta) &\rightarrow -(\tilde{\psi}_\mu, \tilde{\eta}) \\ \chi_{\mu\nu} &\rightarrow -^*\chi_{\mu\nu} . \end{aligned} \tag{1.1}$$

(Here $*$ denotes the Hodge dual: $^*\chi_{\mu\nu} \equiv 1/2 \epsilon_{\mu\nu\rho\sigma} \chi^{\rho\sigma}$; $^{**}\chi_{\mu\nu} = \chi_{\mu\nu}$. The minus signs are because all the fields are in the adjoint representation of the gauge group, represented by antihermitian matrices.)

- $\chi_{\mu\nu}$ **is not self dual.**

Since $\chi_{\mu\nu}$ is not self dual, one might expect its BRST transformation to give the full $F_{\mu\nu}$, so that the path integral would be dominated by flat gauge-field configurations. In fact, this is complicated by the fact that

- **There are 2 vector fields:**

We shall see that it is natural to combine A_μ and V_μ into a complex vector field \mathcal{A}_μ :

$$\begin{aligned} \mathcal{A}_\mu &\equiv A_\mu + i V_\mu \\ \tilde{\mathcal{A}}_\mu &\equiv A_\mu - i V_\mu , \end{aligned} \tag{1.2}$$

and we shall define three different covariant derivatives and field strengths:

$$\begin{aligned} D_\mu X &\equiv \partial_\mu X + [A_\mu, X] , & F_{\mu\nu} &\equiv [D_\mu, D_\nu] \\ \mathcal{D}_\mu X &\equiv \partial_\mu X + [\mathcal{A}_\mu, X] , & \mathcal{F}_{\mu\nu} &\equiv [\mathcal{D}_\mu, \mathcal{D}_\nu] \\ \tilde{\mathcal{D}}_\mu X &\equiv \partial_\mu X + [\tilde{\mathcal{A}}_\mu, X] , & \tilde{\mathcal{F}}_{\mu\nu} &\equiv [\tilde{\mathcal{D}}_\mu, \tilde{\mathcal{D}}_\nu] . \end{aligned}$$

Of course only A_μ is a true connection, since the theory does not have a complexified gauge invariance. Nevertheless, it will turn out that the path integral is dominated by flat complexified connections \mathcal{A}_μ , and the complexified gauge group will play an important role in understanding the ground states of the theory.

2 BRST transformations

In principle, the BRST transformations of the theory could be found by twisting the supersymmetry transformations of the original $N = 4$ Yang–Mills. Instead, we use these transformations and those of the half-twisted theory [5] only as a guide, and fix the transformations by demanding closure of the algebra. For convenience, we define the combinations

$$P^\pm \equiv P \pm i [B, C] . \quad (2.1)$$

The Q BRST transformations are given by:

$$\begin{aligned} Q \mathcal{A}_\mu &= 2i \psi_\mu & Q \tilde{\mathcal{A}}_\mu &= 0 \\ Q \psi_\mu &= 0 & Q \tilde{\psi}_\mu &= \tilde{\mathcal{D}}_\mu B \\ Q C &= i \eta & Q B &= 0 \\ Q \eta &= 0 & Q \tilde{\eta} &= -i P^- \\ Q P^- &= 0 & Q \chi_{\mu\nu} &= \tilde{\mathcal{F}}_{\mu\nu} ; \end{aligned} \quad (2.2)$$

they are clearly nilpotent, with $Q^2 = 0$.

Taking the conjugate of the Q transformations with the hermitian conjugation (1.1), one finds the \tilde{Q} transformations:

$$\begin{aligned} \tilde{Q} \tilde{\mathcal{A}}_\mu &= 2i \tilde{\psi}_\mu & \tilde{Q} \mathcal{A}_\mu &= 0 \\ \tilde{Q} \tilde{\psi}_\mu &= 0 & \tilde{Q} \psi_\mu &= \mathcal{D}_\mu B \\ \tilde{Q} C &= i \tilde{\eta} & \tilde{Q} B &= 0 \\ \tilde{Q} \tilde{\eta} &= 0 & \tilde{Q} \eta &= i P^+ \\ \tilde{Q} P^+ &= 0 & \tilde{Q} \chi_{\mu\nu} &= {}^* \mathcal{F}_{\mu\nu} . \end{aligned} \quad (2.3)$$

These are also nilpotent, with $\tilde{Q}^2 = 0$. This is unlike the WD [1] and YVW [5] theories, in which the BRST transformations close only up to a gauge transformation. However the commutator of Q with \tilde{Q} does give a gauge transformation with parameter $2iB$:

$$\{Q, \tilde{Q}\} = T_{2iB} . \quad (2.4)$$

This commutator also closes on $\chi_{\mu\nu}$ only with the use of its equation of motion. This is a consequence of our inability to find appropriate $B_{\mu\nu}$ auxiliary fields, as we mentioned.

One sees that, as promised, the BRST transforms of $\chi_{\mu\nu}$ give rise to the complexified field strength $\mathcal{F}_{\mu\nu}$, suggesting that the theory is dominated by flat complexified connections. This theory reduces to the Witten–Donaldson theory if one demands reality under

the conjugation (1.1), and takes the BRST operator to be $Q + \tilde{Q}$. Then \mathcal{A}_μ becomes real, only the self-dual part of $\chi_{\mu\nu}$ survives, and the theory reduces to a theory of instantons.

3 The lagrangian

Knowing the BRST transformations, one can now write the most general renormalizable lagrangian with zero ghost number that is invariant under both* Q and \tilde{Q} . As in the untwisted $N = 4$ theory, the lagrangian depends on a coupling constant g and a theta angle† θ . After using ones freedom to rescale fields, while preserving the two BRST transformations of (2.2) and (2.3), one sees that the most general lagrangian depends upon only one more parameter α . It can be written as the sum of three terms:

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{g^2} Q \tilde{Q} \text{Tr} \left(2 \mathcal{D}_\mu C V^\mu + \alpha C P^+ \right) \\ &= \frac{1}{g^2} Q \text{Tr} \left(-2 \mathcal{D}_\mu C \tilde{\psi}^\mu + 2i \mathcal{D}_\mu \tilde{\eta} V^\mu + i \alpha \tilde{\eta} P^+ \right) ;\end{aligned}\quad (3.1)$$

$$\mathcal{L}_2 = Q \text{Tr} \left(-\frac{1}{2g^2} \chi_{\mu\nu} \mathcal{F}^{\mu\nu} + \frac{\theta}{16\pi^2} {}^* \chi_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu} \right) ; \quad (3.2)$$

$$\mathcal{L}_3 = \frac{2i}{g^2} \text{Tr} \left({}^* \chi^{\mu\nu} \tilde{\mathcal{D}}_\mu \tilde{\psi}_\nu - \frac{1}{4} B \left\{ {}^* \chi^{\mu\nu}, \chi_{\mu\nu} \right\} \right) . \quad (3.3)$$

Evaluating the BRST transformations, one finds:

$$\begin{aligned}\mathcal{L} &= \frac{1}{g^2} \text{Tr} \left[-\frac{1}{2} \mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu} + \alpha \left(P - \frac{1}{\alpha} D_\mu V^\mu \right)^2 - \frac{1}{\alpha} (D_\mu V^\mu)^2 \right. \\ &\quad - \left(\mathcal{D}_\mu C \tilde{\mathcal{D}}^\mu B + \tilde{\mathcal{D}}_\mu C \mathcal{D}^\mu B \right) + \alpha [B, C]^2 \\ &\quad + 2i \tilde{\psi}^\mu \mathcal{D}_\mu \eta + 2i \psi^\mu \tilde{\mathcal{D}}_\mu \tilde{\eta} + 2i \chi^{\mu\nu} \mathcal{D}_\mu \psi_\nu + 2i {}^* \chi^{\mu\nu} \tilde{\mathcal{D}}_\mu \tilde{\psi}_\nu \\ &\quad \left. - 2i \alpha B \left\{ \eta, \tilde{\eta} \right\} + 4i C \left\{ \psi_\mu, \tilde{\psi}^\mu \right\} - \frac{i}{2} B \left\{ \chi_{\mu\nu}, {}^* \chi^{\mu\nu} \right\} \right] \\ &\quad + \frac{\theta}{16\pi^2} \text{Tr} \left(\tilde{\mathcal{F}}_{\mu\nu} {}^* \tilde{\mathcal{F}}^{\mu\nu} \right) .\end{aligned}\quad (3.4)$$

*If we were to have demanded invariance just under Q , and not also under \tilde{Q} , the only extra generality that we would have had would have been the freedom to have altered two of the coefficients in the four terms of \mathcal{L}_1 in (3.1). (Recall from (2.1) that P^+ contains two terms.) We shall not use this extra freedom, since such a change would be ugly and, being BRST exact, would not affect any of our results.

†One might naively have expected to have three independent theta angles, corresponding, say, to $\int F \wedge F$, $\int \mathcal{F} \wedge \mathcal{F}$ and $\int \tilde{\mathcal{F}} \wedge \tilde{\mathcal{F}}$. However, since only A is a true gauge field, all of these define the same characteristic class: since $F \wedge F$ is locally the exterior derivative of a Chern–Simons term, and the difference between the Chern–Simons terms of A and of \mathcal{A} is globally defined, all three terms are equal.

(Here we have integrated by parts to isolate the equation of motion of the auxiliary field P .) The untwisted $N = 4$ theory corresponds to $\alpha = 1$, in which case the kinetic term of the V_μ 's is simply proportional to $V_\mu \square V^\mu$. Important features of the lagrangian include:

- **Q invariance:**

While \mathcal{L}_1 and \mathcal{L}_2 are manifestly exact under Q , \mathcal{L}_3 is not exact. However, by using the Jacobi identity and the definition of the field strength, one easily sees that \mathcal{L}_3 , and therefore \mathcal{L} , is invariant under Q .

- **\tilde{Q} invariance:**

It is less clear that \mathcal{L} is invariant under \tilde{Q} . However \mathcal{L} is invariant under Q , and is real with respect to the hermitian conjugation operation (1.1), so it is also invariant under \tilde{Q} .

- **topological nature:**

\mathcal{L}_3 is not Q -exact, but since it is a four form, its integral is independent of the metric. Thus the full *stress tensor* of the theory is Q -exact. This is the definition of a topological field theory, and is sufficient to guarantee that physical quantities have no dependence on the metric of the underlying four-manifold[‡].

- **scale, but not conformal invariance:**

The stress tensor of the theory is Q -exact, but it does not vanish. Therefore, as in the WD theory [1] and in the other $N = 4$ twistings [5], the topological theory is *not* conformally invariant. However, after using the equations of motion, the trace of the stress tensor becomes the divergence of a current:

$$T_\mu^\mu \sim \frac{4}{g^2} \partial_\mu \text{Tr} \left[C D^\mu B - \frac{1}{\alpha} (D \cdot V) V^\mu + i \eta \tilde{\psi}^\mu + i \tilde{\eta} \psi^\mu \right], \quad (3.5)$$

so one does have a global scale (or Weyl) invariance [1]. This is easy to understand without having to have done this calculation, since BRST invariance means that one can not add curvature terms, mass terms or cubic scalar couplings to the original scale-invariant $N = 4$ Yang–Mills. The scale invariance holds at the full quantum level, since $N = 4$ Yang–Mills is finite.

- **coupling independence:**

In many ways, α appears like a gauge-fixing parameter in the lagrangian, suggesting the gauge fixing of a complexified gauge symmetry. Terms involving it are Q -exact (*c.f.* (3.1)), and physical quantities should not depend on it. Unlike the other topological Yang–Mills theories, the instanton-number term is also Q -exact (*c.f.* (3.2)), so there should also

[‡]If one had had the appropriate auxiliary fields, the entire lagrangian would presumably have been $Q \tilde{Q}$ exact. Such a situation occurs in the 2-dimensional B-twisted topological sigma model of [8], which was written with auxiliary fields in [9].

be no dependence on θ . (This apparently surprising result is actually pretty reasonable when one realizes that the path-integral is dominated by flat connections, which have zero instanton number.) While it is less obvious, the same is also true of the coupling constant g . In addition to its appearance in exact terms, g appears only in the factor in front of \mathcal{L}_3 . This dependence can be removed without disturbing the Q BRST transformation by rescaling $\tilde{\psi}$ and B by a factor of g^2 , and C and η by g^{-2} . We do not perform such a rescaling, since it obscures the symmetry between Q and \tilde{Q} , but its existence shows that physical quantities also should not depend on g . Once again, this conclusion should be obvious in a formulation with the appropriate auxiliary fields.

We conclude that *any amplitude of the theory should be a pure number, depending only on the gauge group and the topology of the manifold.*

4 Ground states: The geometry of the moduli space

Since the theory is independent of the gauge coupling, one can see what it studies by examining it in the weak coupling limit, where it is dominated by its ground states [1]. As in the WD theory, one first needs to “Wick rotate” $C \rightarrow -B^\dagger$. (This is incompatible with our hermiticity operation (1.1), but that will not cause us any problems.) After integrating out the auxiliary field P , the purely bosonic part of the action can be written as the sum of positive semi-definite terms, plus the topological theta term:

$$\begin{aligned} \mathcal{S}_B = & \frac{1}{g^2} \left(\frac{1}{2} \left\| \mathcal{F}_{\mu\nu} \right\|^2 + \frac{1}{\alpha} \left\| D_\mu V^\mu \right\|^2 \right. \\ & \left. + \left\| \tilde{\mathcal{D}}^\mu B \right\|^2 + \left\| \mathcal{D}^\mu B \right\|^2 + \alpha \left\| [B, B^\dagger] \right\|^2 \right) \\ & + \frac{\theta}{16\pi^2} \int \text{Tr} \left(\tilde{\mathcal{F}}_{\mu\nu}^* \tilde{\mathcal{F}}^{\mu\nu} \right), \end{aligned} \quad (4.1)$$

where $\|X\|^2$ means the integral of $\text{Tr} X X^\dagger$ over the manifold, soaking up indices with the spacetime metric where needed. As $g \rightarrow 0$, the path integral is clearly dominated by configurations which are at the absolute minimum of the action, within each topological class. This is attained when all the objects in all the norms in (4.1) vanish identically. Note that since such configurations are flat ($\mathcal{F}_{\mu\nu} = 0$), the theta term does not contribute, as expected, so $\mathcal{S}_B = 0$ at the ground states. In a topological theory, one can formally argue that the path integral is dominated by the fixed points of the BRST transformation [3]. Examining the the Q and \tilde{Q} transformations in (2.2) and (2.3), one sees that this agrees with the condition that \mathcal{S}_B vanishes.

The first condition for minimizing the action, $\mathcal{F}_{\mu\nu} = 0$, has a clear geometric meaning,

but the meaning of the other conditions may seem rather less transparent. Following the logic of Vafa and Witten in the YVW twisting, one can look for “vanishing theorems” to eliminate these [7]. Thus the scalar field B is a kind of bosonic partner of the ghosts used for fixing the gauge invariance of the theory, and its treatment is very similar to that in the WD and YVW theories. The only slight complication is that there are two connections in our case. Its minimization condition is

$$\tilde{\mathcal{D}}_\mu B = \mathcal{D}_\mu B = [B, B^\dagger] = 0, \quad (4.2)$$

with the hermitian-conjugate equations for B^\dagger . This means that B and B^\dagger can be simultaneously diagonalized by a gauge transformation, and that their \mathcal{D}_μ and $\tilde{\mathcal{D}}_\mu$ derivatives both vanish. Any solution to (4.2) with non-zero B implies that the connection \mathcal{A}_μ is reducible, and such configurations cause trouble, since they imply that the moduli space is not compact. In Donaldson theory one usually restricts oneself to the gauge groups $SU(2)$ and $SO(3)$, so any reducible connection is either trivial, or lies completely in a $U(1)$ subgroup. The latter are then discarded by considering only manifolds with vanishing first Betti number b_1 . In section 5 we shall see that this theory is nontrivial only on manifolds with $b_1 \geq 1$, so we shall have no choice but to deal with these reducible connections.

Continuing in this way, one could also try to find a vanishing theorem for V_μ on appropriate manifolds, as Vafa and Witten did for the antisymmetric tensor field B_{ij}^+ in their twisting [7]. This can be done because, if one “chooses the Feynman gauge” $\alpha = 1$ to simplify the V_μ kinetic term, an integration by parts leads to a miraculous cancellation of the terms in the lagrangian linear in $F_{\mu\nu}$. Dropping the instanton number term, the action for A_μ and V_μ becomes:

$$g^2 \mathcal{S}_{A,V} = \frac{1}{2} \|F_{\mu\nu}\|^2 + \|D_\mu V_\nu\|^2 + \frac{1}{2} \| [V_\mu, V_\nu] \|^2 - \int R_{\mu\nu} \text{Tr } V^\mu V^\nu. \quad (4.3)$$

Recalling that V_μ is *anti*-hermitian, one sees that if the Ricci tensor of the manifold is strictly positive definite as a matrix, then V_μ must vanish at a minimum of the action $\mathcal{S}_{A,V} = 0$. Note that applying the same logic to the $U(1)$ case leads to the well-known result that harmonic 1-forms on such manifolds must vanish, so they have $b_1 = 0$. If the Ricci tensor is positive *semi*-definite—in particular if it vanishes—one sees that all the components of a non-zero V_μ at a minimum of the action commute, and so can be simultaneously diagonalized by a gauge transformation. Since V_μ is also covariantly constant, the gauge field is again reducible. Examining the $U(1)$ case, one now finds the very strong constraint that any harmonic 1-form on such a manifold must also be a Killing vector. This means that either $b_1 = 0$ again, so V_μ must vanish, or that, at least locally, \mathcal{M} is simply the product of a circle with a lower dimensional manifold.

However, since this theory has nothing to calculate on manifolds with $b_1 = 0$, we shall need to understand the geometric meaning of the full moduli space with non-zero V_μ , defined by the two equations

$$\mathcal{F}_{\mu\nu} = \tilde{\mathcal{F}}_{\mu\nu} = 0 \quad (4.4a)$$

and

$$D \cdot V = \frac{i}{2} [\mathcal{D}_\mu, \tilde{\mathcal{D}}^\mu] = 0 . \quad (4.4b)$$

We have suggested that in many ways \mathcal{A}_μ acts as the connection of a complexified gauge group $G^\mathbb{C}$. While it is impossible to properly implement this idea in the action itself, it is very important in understanding the moduli space of the theory. The flatness condition (4.4a) is indeed invariant under $G^\mathbb{C}$ transformations. Imposing $D \cdot V = 0$ then looks like a partial gauge fixing of $G^\mathbb{C}$ to the ordinary gauge group G . More precisely, our claim is that the space of irreducible solutions of (4.4), modulo G , is equivalent to the space of irreducible solutions to $\mathcal{F}_{\mu\nu} = 0$, modulo $G^\mathbb{C}$. In other words, the moduli space is the space of flat connections of the complexified gauge group.

Trivially, any point in the moduli space (4.4) gives a single point in the moduli space of flat complexified connections. The problem is to show that the $G^\mathbb{C}$ -orbit of any flat complexified connection contains one and only one point on the original moduli space. This is very similar to the Gribov problem of choosing the coulomb gauge for an ordinary gauge field, and, in order to argue for the *uniqueness* of a solution to (4.4b), one can modify a technique used there. Thus, consider the orbit of any (not necessarily flat) connection under *complexified* gauge transformations $\Omega \in G^\mathbb{C}$, with $\mathcal{D}_\mu^\Omega \equiv \Omega^{-1} \mathcal{D}_\mu \Omega$, and define the real, positive semi-definite function

$$F[\Omega] = -\frac{1}{2} \text{Tr} \int g^{\mu\nu} V_\mu^\Omega V_\nu^\Omega \quad (4.5)$$

on that orbit. For an infinitesimal $\Omega \rightarrow 1 + \omega$, one finds

$$\delta F = i \text{Tr} \int g^{\mu\nu} V_\mu D_\nu \omega^+ , \quad (4.6)$$

where $\omega^+ \equiv (\omega + \omega^\dagger)/2$ is the part of ω in the algebra of the coset $G^\mathbb{C}/G$. Thus F has an extremum exactly when (4.4b) is satisfied. At second order one finds

$$\delta^2 F = \left\| \mathcal{D}_\mu \omega^+ \right\|^2 , \quad (4.7)$$

so any such extremum is a strict minimum (except for the pure gauge directions), if the connection is irreducible. This means that $F[\Omega]$ is a strictly convex Morse function, and

the minimum must be unique*. (This proof is a particular example of a result of Kempf and Ness [10].)

The problem is that, while it may be plausible, we have not proven that the minimum of $F[\Omega]$ actually *exists*. This can be done using “moment-maps”[†]. (We actually need a generalization of this idea to infinite-dimensional spaces, as used, for example, in Atiyah and Bott’s study of the moduli space of Yang–Mills theories on a Riemann surface [11].) First note that the space of complexified connections has a natural symplectic structure, with A_μ and V_ν satisfying the Poisson bracket

$$\{A_\mu, V_\nu\} = g_{\mu\nu} . \quad (4.8)$$

Using this bracket, one easily sees that (ordinary) gauge transformations are generated by the moment map $\mu(A, V) = D \cdot V$, which is just our familiar “gauge-fixing condition” (4.4b). Guillemin and Sternberg showed that the quotient of the full space by $G^\mathbb{C}$ is equivalent to the quotient of $\mu^{-1}(0)$ by G [12]. In other words, (4.4b) uniquely fixes $G^\mathbb{C}$ to G , and our moduli space is indeed the space of flat complexified connections divided by complexified gauge transformations.

5 Operators, anomalies and amplitudes

Now, in order to study the amplitudes in a topological field theory, one needs to find the operators in the BRST cohomology, and to determine the ghost-number anomaly of the theory. In the WD theory the cohomology starts with “instanton number” 4-form $F \wedge F$, with ghost number 0. Following the usual procedure, this descends to give a 3-form, with ghost number 1, and so on until one reaches a 0-form with ghost number 4 [1]. The ghost-number anomaly depends on the instanton number k , and the Euler number χ and signature σ of the manifold. For $SU(2)$ one has

$$\Delta Q_{GH}(\text{WD}) = 8k - \frac{3}{2}(\chi + \sigma) = 8k - 3(1 - b_1 + b_2^+) . \quad (5.1)$$

*One can also show the *uniqueness* of any irreducible solution directly (This proof is due to Ori Ganor): Assume that one has one solution \mathcal{D}_μ to (4.4), and substitute \mathcal{D}_μ^Ω into the “gauge condition” (4.4b). Multiplying on the left by Ω , and on the right by Ω^\dagger , one finds $g^{\mu\nu} M \tilde{\mathcal{D}}_\nu (M^{-1} \mathcal{D}_\mu M) = 0$, where $M \equiv \Omega \Omega^\dagger$. Integrating this over the manifold, and doing an integration by parts, one sees that $\mathcal{D}_\mu M$ necessarily vanishes. The crucial difference from the Gribov case is that here M^{-1} is a hermitian matrix. The irreducibility of the connection now implies that M is in the trivial representation of G , and so is proportional to the unit matrix. The only possibility is $M = \Omega \Omega^\dagger = 1$, meaning that Ω is an ordinary G transformation. Thus the solution to (4.4b) is unique.

[†]I would like to thank Jeremy Schiff for introducing me to moment maps, and for pointing out that $D \cdot V$ acts like one.

This is generically nonvanishing, and it gives the (virtual) dimension of the moduli space of instantons on the manifold. The expectation value of sets of operators—in practice of 2-forms—that balance the anomaly then give the Donaldson invariants of the manifold [1]. One has the same cohomology, but a different anomaly, in the WD theories with extra matter [5, 6].

In the YVW theory, one needs to arbitrarily pick one of the two BRST operators of the theory, say Q^1 . Its cohomology is then the same as in the WD theory. (These will not be in the cohomology of Q^2 .) The ghost number is actually part of an unbroken $SU(2)$, so it has no anomaly. Since all the operators in cohomology have positive ghost number, the only nonvanishing amplitude of the theory is its partition function. Barring holomorphic anomalies, this is a holomorphic function of $\tau \equiv \theta/2\pi$, with interesting properties under modular transformations of τ , coming from S-duality [7].

In our case one might think that the BRST cohomology would again be a copy of the WD cohomology. Indeed the WD ghost-number 4 scalar operator

$$O^{(0)} = \text{Tr} \left(B^2 \right) \quad (5.2)$$

is in the cohomology of both Q and \tilde{Q} . However, here its descent is somewhat unusual. One first finds a 1-form in the Q -cohomology, say, which is \tilde{Q} exact:

$$O^{(1)} = \text{Tr} \left(B \tilde{\psi} \right) = \tilde{Q} \text{Tr} \left(B V \right) . \quad (5.3)$$

This then descends to a 2-form

$$O^{(2)} = \text{Tr} \left(B^* \mathcal{F} - i \tilde{\psi} \wedge \tilde{\psi} \right) = \tilde{Q} \text{Tr} \left(B \chi - i V \tilde{\psi} \right) , \quad (5.4)$$

which is not only \tilde{Q} exact, but which also closes only with the use of the χ equation of motion. The procedure then stops, and one does not find any 3- or 4-forms in the cohomology. In particular, we already know that the instanton number 4-form is exact. We have not found any other operators in the BRST cohomology.

All the twisted $N = 4$ theories can have at most purely gravitational anomalies, since their ghost number symmetries are subgroups of the $SU(4)$ R-parity of the original $N = 4$ theory, which is anomaly free. Thus the most simple way to calculate the anomaly in this theory would be to turn off all the bosonic fields. For the moment, we shall prefer to work in the more general background of an irreducible flat connection. The covariantized exterior derivative \mathcal{D} then squares to 0, so one has a twisted elliptic complex

$$0 \longrightarrow \Omega^0 \xrightarrow{\mathcal{D}} \Omega^1 \xrightarrow{\mathcal{D}} \Omega^2 \xrightarrow{\mathcal{D}} \Omega^3 \xrightarrow{\mathcal{D}} \Omega^4 \longrightarrow 0 , \quad (5.5)$$

where Ω^n denotes the space of n -forms in the adjoint of the complexified gauge group $G^{\mathbb{C}}$. Recalling that B and B^\dagger vanish in these backgrounds, one can see that the zero modes of $\eta, \psi, {}^*\chi, {}^*\tilde{\psi}$ and ${}^*\tilde{\eta}$ are the harmonic representatives of the cohomology classes $H_{\mathcal{A}}^n$ of this complex, for $n = 0 \dots 4$. For example, by considering the equations of motion involving ψ , one can see that it satisfies

$$\mathcal{D}\psi = \tilde{\mathcal{D}} * \psi = 0 . \quad (5.6)$$

Because the ghost numbers of these fermions alternate between -1 and 1 , the anomaly is minus the index of this complex. To calculate this one can now turn \mathcal{A} off, so the index is simply the dimension of G times the index of the de Rham complex. Thus*

$$\Delta Q_{GH} = -\chi \dim G = \dim G (2b_1 - 2 - b_2) . \quad (5.7)$$

This anomaly has dramatic consequences: Since all of the operators in the BRST cohomology have strictly positive ghost number, the theory can have nontrivial amplitudes only over manifolds with zero or negative Euler number. Such manifolds have $b_1 \geq 1$, meaning that one will be forced to consider $U(1)$ -reducible connections.

Since $O^{(1)}$ and $O^{(2)}$ are \tilde{Q} -exact, and all the operators in the Q cohomology are closed under \tilde{Q} , any amplitude involving them will vanish. Thus one needs only to calculate amplitudes of $O^{(0)}$'s. The explicit form of $O^{(0)}$ in (5.2) is unique only for $SU(2)$ or $SO(3)$, which are the groups usually studied in Donaldson theory. In those cases $O^{(0)}$ has ghost number 4, and there is a unique nonvanishing amplitude[†]

$$\Omega = \left\langle \prod_{i=1}^{-\chi \dim G / 4} \text{Tr} (B^2(x_i)) \right\rangle , \quad (5.8)$$

where the x_i 's are arbitrarily chosen points on the manifold. Note, however, that this amplitude exists only if the Euler number of the manifold is divisible by 4. To find a nontrivial invariant on other manifolds, one needs to consider larger gauge groups; there will then be several Ω 's corresponding to various combinations of the casimirs of the groups. In this case, the theory has the additional complication that there may also be *nonabelian* reducible connections.

*This is the usual anomaly that one would expect in a theory of flat connections [13]. The only surprising feature of our construction is that we need a complex with complexified connections. It is interesting that if one tries to construct a topological field theory of flat connections on G directly in four dimensions, as has been done in two and three dimensions [14], one is again forced to consider complexified connections [15].

[†]For $SO(3)$ one needs to sum over bundles with all possible second Stiefel–Whitney classes w_2 [7]. Since Ω is a pure number, S-duality should imply that it is the same for $SU(2)$ and for $SO(3)$.

6 Calculating the invariant Ω

In order to see what Ω actually calculates, let us assume for the moment that we are at a point at which there are no fermionic zero modes. (Thus we are considering a manifold with vanishing Euler number, and Ω reduces to the partition function of the theory.) The moduli space now consists of isolated points, so one only needs to see what each point contributes to the partition function [7]. Noting that \mathcal{A}_μ is the only field with a nontrivial background in this case, we expand

$$\mathcal{A}_\mu \rightarrow \mathcal{A}_\mu + a_\mu . \quad (6.1)$$

Now, in order to evaluate the path integral, one must fix the gauge. Since \mathcal{D}_μ and $\tilde{\mathcal{D}}^\mu$ commute (4.4b), there is a unique “scalar laplacian” $\square_{\mathcal{A}} \equiv \mathcal{D}_\mu \tilde{\mathcal{D}}^\mu$ at any ground state of the theory. We thus choose an obvious generalization of the Lorentz gauge-fixing*,

$$\mathcal{G} = \frac{1}{2} \left(\tilde{\mathcal{D}}_\mu a^\mu + \mathcal{D}_\mu \bar{a}^\mu \right) , \quad (6.2)$$

which transforms as $\delta \mathcal{G} = \square_{\mathcal{A}} \lambda$ under an infinitesimal gauge transformation λ . We then follow the usual Faddeev–Popov procedure (in Feynman gauge), and get the semiclassical gauge-fixed lagrangian by expanding to quadratic order in the quantum fields:

$$\begin{aligned} g^2 \mathcal{L}_{\text{quad}} \rightarrow \text{Tr} \left[a_\mu \left(g^{\mu\nu} D^\rho \tilde{\mathcal{D}}_\rho + [\tilde{\mathcal{D}}^\mu, \mathcal{D}^\nu] \right) \bar{a}_\nu + 2C \square_{\mathcal{A}} B + 2b \square_{\mathcal{A}} c \right. \\ \left. + 2i \tilde{\psi}^\mu \mathcal{D}_\mu \eta + 2i \psi^\mu \tilde{\mathcal{D}}_\mu \bar{\eta} + 2i \chi^{\mu\nu} \mathcal{D}_\mu \psi_\nu + 2i {}^* \chi^{\mu\nu} \tilde{\mathcal{D}}_\mu \tilde{\psi}_\nu \right] . \end{aligned} \quad (6.3)$$

Here c is the (anti-commuting) ghost and b the antighost. Recall that this semiclassical approximation becomes exact as $g \rightarrow 0$.

The functional integral over B and C now simply cancels that over b and c . The integral over a_μ and \bar{a}_ν gives $\det^{-1}(-\square_{\mathcal{A}}^{\mu\nu})$, where $\square_{\mathcal{A}}^{\mu\nu}$ is the hermitian negative-semi-definite “vector laplacian” $g^{\mu\nu} D^\rho \tilde{\mathcal{D}}_\rho + [\tilde{\mathcal{D}}^\mu, \mathcal{D}^\nu]$. This laplacian has no zero-modes at an isolated point in the moduli space. The integral over the fermionic fields gives the Pfaffian of their kinetic operators, which is convenient studied by squaring the operators, and then taking the square root. ψ and $\tilde{\psi}$ satisfy the squared equations of motion:

$$\begin{aligned} D^\rho \tilde{\mathcal{D}}_\rho \psi_\mu + [\tilde{\mathcal{D}}^\mu, \mathcal{D}^\nu] \psi_\nu &= \square_{\mathcal{A}}^{\mu\nu} \psi_\nu = 0 \\ D^\rho \tilde{\mathcal{D}}_\rho \tilde{\psi}_\mu + [\mathcal{D}^\mu, \tilde{\mathcal{D}}^\nu] \tilde{\psi}_\nu &= \square_{\mathcal{A}}^{T\mu\nu} \tilde{\psi}_\nu = 0 , \end{aligned} \quad (6.4)$$

where T indicates the operator transpose. Thus, under our assumption that there are no fermionic zero modes, the square of the Pfaffian is $\det^2 \square_{\mathcal{A}}^{\mu\nu}$, and one sees explicitly that

*In terms of A_μ and V_μ , this is $\mathcal{G} = D_\mu \delta A^\mu + [V_\mu, \delta V^\mu]$.

the Pfaffian cancels the bosonic determinant up to a sign, as must occur in a topological theory. Because each eigenvalue of the fermion kinetic operators occurs twice, once for ψ and once for $\tilde{\psi}$, the sign must be positive. Thus, in this simple case the partition function simply counts the number of points in the moduli space of the theory. This is like the YVW twisting, and unlike the WD theory, in which the partition function becomes a *signed* sum over the points in the moduli space [7].

On a manifold with negative Euler number one will necessarily have zero modes of ψ_μ and $\tilde{\psi}_\mu$. For simplicity, we shall still restrict ourselves to irreducible connections, so there are no η or $\tilde{\eta}$ zero modes, and we shall *assume* that there are no $\chi_{\mu\nu}$ zero modes[†]. From the index theorem (see (5.7)), this means that there are $(-\chi) \cdot \dim G/2$ zero modes each of ψ_μ and $\tilde{\psi}_\mu$, so that the moduli space is a complex manifold of dimension $(-\chi) \cdot \dim G/2$. The calculation of Ω could be carried out straightforwardly, by noting that these zero modes are soaked up by bringing down cubic $C\psi\tilde{\psi}$ terms from the lagrangian, and then using Wick's theorem to get B - C propagators, but a simpler method is to note that one can use the equation of motion of C in the semiclassical approximation [13]. Thus, one integrates

$$\square_A B + 2i \{ \psi_\mu, \tilde{\psi}^\mu \} = 0 \quad (6.5)$$

to find B , substitutes this into the definition of Ω in (5.8), and integrates the result over the zero modes of a_μ , ψ_μ and their conjugates. Only the zero-mode parts of ψ_μ and $\tilde{\psi}_\mu$ in (6.5) contribute, and once one has integrated over them, Ω becomes the integral of a $(-\chi) \cdot \dim G$ form over the moduli space of flat connections.

Of course, the procedure that we have described is rather formal, and we should state some caveats: On any manifold where Ω does not trivially vanish by ghost-number conservation, there will necessarily be $U(1)$ -reducible connections; in addition, there may be points on the moduli space where H_A^2 is nonvanishing. The moduli space will be rather singular at both these sorts of points, and one will have to deal with these singularities in order for Ω to be well defined. Note also that the identification of the moduli space of the theory with that of complexified flat connections fails when connections are reducible. An additional problem that is specific to this particular theory, is that the moduli space may be noncompact because of the non-compactness of the complexified gauge group $G^\mathbb{C}$. Finally, one should check that BRST-exact variations of Ω do not give rise to contributions from the boundaries or from the singular regions of moduli space. All of these issues must be dealt with in order to make Ω well-defined.

[†]We have not been able to find any vanishing theorem for $\chi_{\mu\nu}$ in our case, since the terms linear in the field strength do not cancel upon an integration by parts. It will be important to know in what cases such zero modes do or do not occur.

7 Discussion

We have studied a twisting of $N = 4$ Yang–Mills which leads to a new type of topological field theory (TFT). The theory is defined on any smooth orientable compact four-manifold \mathcal{M} , with any semisimple gauge group G . Unlike the other topological Yang–Mills theories, which are dominated by instanton configurations, the path integral here is dominated by flat connections \mathcal{A}_μ in the *complexified gauge group* $G^\mathbb{C}$. Because of the ghost-number anomaly of the theory, and the structure of its BRST cohomology, the theory is trivial when the Euler number χ of the four-manifold \mathcal{M} is positive. Otherwise it gives rise to a single amplitude Ω , defined in eq. (5.8), which is the integral of a $(-\chi) \cdot \dim G$ form over the moduli space of flat complexified connections on \mathcal{M} .

The moduli space of flat connections on \mathcal{M} —let us call it \mathcal{C} —depends only on the gauge group and the fundamental group π_1 of \mathcal{M} . It is given by homeomorphisms of π_1 to G (given by integrating Wilson lines of the connection over nontrivial cycles), modulo conjugation (gauge transformations): $\mathcal{C} = \text{Hom}(\pi_1, G)/G$ [13]. Thus any well-defined quantity on \mathcal{C} is obviously a topological invariant of the manifold, and is even invariant under homomorphisms that are not diffeomorphisms. Finding such invariants is orthogonal to the Freedman program of classifying four-manifolds [16] (and the Donaldson program of classifying smooth four-manifolds [4]), in which \mathcal{M} is usually taken to be simply connected—If $\pi_1 = 0$, \mathcal{C} contains only the trivial connection.

This means that Ω is indeed a topological invariant of \mathcal{M} . However, it is natural to ask “Why does one need to deal with the complexified group?”, and “What is so special about Ω ?”. In other words, “What has this TFT given us?”. If one knows the fundamental group of a manifold then \mathcal{C} is completely determined, and there is no need to try to solve for Ω . However, π_1 is notoriously difficult to find. One could hope that since Ω , and Ω alone, comes from a TFT, it would be the simplest object on \mathcal{C} that one could actually calculate. This is not really possible using the straightforward techniques of our paper, in which one first has to directly construct \mathcal{C} . However, it may be worth noting that the reformulation of the Donaldson theory in terms of a TFT did not lead to practical methods for calculating invariants, until the techniques of Seiberg and Witten [17] were used to study that theory in its infra-red limit, where it reduces to a much simpler abelian theory of “monopoles” [18]. This particular technique is not helpful in our theory, since it is completely scale invariant at the quantum level. (This is also the case for the YVW theory that calculates the Euler number of the moduli space of instantons.) Nevertheless, we can hope that new techniques, possibly coming from the close association of $N = 4$ Yang–Mills with string theory, will become available to study this theory. Even if this

does not occur, one knows at the very least that Ω will satisfy cutting relations, simply because it comes from a TFT. This is presumably not the case for other invariants on \mathcal{C} . Thus one should be able to calculate the Ω 's of manifolds formed by surgery.

It is interesting to compare this situation with that in three dimensions. One can define the ‘‘Casson invariant’’ λ on oriented three-manifolds using a cutting procedure [19]. Taubes has shown that, at least on homology three spheres*, λ is the Euler number of flat $SU(2)$ connections [22], and Witten has pointed out that this is given by the partition function of a certain three-dimensional topological field theory [23, 14, 24]. There is an obvious analogy between Ω and λ . The Casson invariant has been very useful in classifying topological three-manifolds, and we hope that Ω could be useful in the four-dimensional case.

Acknowledgments

I would like to thank Maxim Braverman, Jeremy Schiff, Ori Ganor and Michael Farber for their patience in many long and helpful discussions. I am also grateful to Cobi Sonnenschein, Yaron Oz, Ben Svetitsky and Alyosha Morozov for helpful comments.

* λ was originally defined only for homology three spheres in [19]; this was extended to rational homology three spheres (*i.e.* three-manifolds with $b_1 = 0$) in [20]. On such manifolds the moduli space ‘‘generically’’ consists of a discrete number of points [21], and one evades the issue of how to deal with the $U(1)$ -reducible connections.

References

- [1] E. Witten, *Comm. Math. Phys.* **117** (1988) 353.
- [2] E. Witten, *Comm. Math. Phys.* **118** (1988) 411, *Nucl. Phys.* **B340** (1990) 281;
T. Eguchi and S.-K. Yang, *Mod. Phys. Lett.* **A5** (1990) 1693;
R. Brooks and D. Kastor, *Phys. Lett.* **246B** (1990) 99.
- [3] E. Witten, *Nucl. Phys.* **B371** (1992) 191.
- [4] S. Donaldson, *Topology* **29** (1990) 257.
- [5] J.P. Yamron, *Phys. Lett.* **213B** (1988) 325.
- [6] D. Anselmi and P. Fré, *Nucl. Phys.* **B404** (1993) 288, hep-th/9211121; *Nucl. Phys.* **B416** (1994) 255, hep-th/9306080; *Phys. Lett.* **347B** (1995) 247, hep-th/9411205;
M. Alvarez and J.M.F. Labastida, *Phys. Lett.* **315B** (1993) 251, hep-th/9305028; *Nucl. Phys.* **B437** (1995) 356, hep-th/9404115; “Non-abelian monopoles on four-manifolds”, preprint US-FT-4/95, hep-th/9504010.
S. Hyun, J. Park and J.-S. Park, “Topological QCD_c”, preprint YUMS-95-08, CALT-68-1985, SWAT/67, hep-th/9503201.
- [7] C. Vafa and E. Witten, hep-th/9408074, *Nucl. Phys.* **B431** (1994) 3.
- [8] E. Witten, “Mirror manifolds and topological field theory”, in “Essays on Mirror Manifolds”, ed. S.T. Yau (International Press, 1992), hep-th/9112056.
- [9] J.M.F. Labastida and M. Mariño, hep-th/9405151, *Phys. Lett.* **333B** (1994) 386.
- [10] G. Kempf and L. Ness, in “Algebraic Geometry”, *Lect. Notes in Math.* **732** (1979) 233 (Springer-Verlag, 1979).
- [11] M.F. Atiyah and R. Bott, *Philos. Trans. R. Soc. Lond.* **A308** (1982) 523.
- [12] V. Guillemin and S. Sternberg, *Invent. Math.* **67** (1982) 491; *ibid.* 515.
- [13] See, for example, D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rept.* **209C** (1991) 129.
- [14] J. Sonnenschein, *Phys. Rev.* **D42** (1990) 2080.
- [15] J. Sonnenschein, private communication.
- [16] M.H. Freedman, *J. Diff. Geom.* **17** (1982) 357.
- [17] N. Seiberg and E. Witten, hep-th/9407087, *Nucl. Phys.* **B426** (1994) 19, erratum *ibid.* **430** (1994) 485; hep-th/9408099, *Nucl. Phys.* **B431** (1994) 484.

- [18] E. Witten, “*Monopoles and four manifolds*”, IAS preprint IASSNS-HEP-94-96, hep-th/9411102.
- [19] A. Casson, “*An invariant for homology 3-spheres*”, Lectures at MSRI, Berkeley (1985).
- [20] K. Walker, “*An extension of Casson’s invariant*”, (Princeton University Press, 1992).
- [21] R. Fintushel and R.J. Stern, in “*Geometry of low-dimensional manifolds: 1*”, ed. S.K. Donaldson and C.B. Thomas (Cambridge University Press, 1990).
- [22] C.H. Taubes, *J. Diff. Geom.* **31** (1990) 547.
- [23] E. Witten, *Nucl. Phys.* **B323** (1989) 113.
- [24] M. Blau and G. Thompson, hep-th/9112012, *Comm. Math. Phys.* **152** (1993) 41.